Garside structures for $B(e, e, n)$

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Complex reflection groups

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Fact: Every reflection group is a direct product of irreducible ones.

Irreducible reflection groups

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The first family:
For $e, n \geq 1$, $G(e, e, n)$ is the group of $n \times n$ matrices consisting of:

- monomial matrices (each row and column has a unique nonzero entry),
- with all nonzero entries lying in μ_e, the e-th roots of unity, and
- for which the product of the nonzero entries is 1.

Irreducible reflection groups

The second family:
For $e, n \geq 1$, $G(2e, e, n)$ is the group of $n \times n$ matrices consisting of:

- monomial matrices,
- with all nonzero entries lying in μ_{2e}, and
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Plus: 15 exceptions!

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$$\mathcal{A} = \{Ker(s-1) \mid s \in \mathcal{R}\},\ X = \mathbb{C}^n \setminus \cup\mathcal{A}$$

- $P = \pi_1(X)$ pure (complex) braid group,
- $B = \pi_1(X/W)$ braid group, with

$$1 \longrightarrow P \longrightarrow B \longrightarrow W \longrightarrow 1$$

Artin-Tits groups

A finite Coxeter group W is defined by the following presentation:

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The Artin-Tits group $B(W)$ attached to W is the group of fractions of the monoid $B^+(W)$:

$$B^+(W) = < \widetilde{S} \mid \underbrace{\tilde{s}\tilde{t}\tilde{s}\cdots}_{m_{st}} = \underbrace{\tilde{t}\tilde{s}\tilde{t}\cdots}_{m_{st}} \text{ for } \tilde{s} \neq \tilde{t} \in \widetilde{S} >^+$$

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W a real reflection group

(Coxeter) $\updownarrow$

W is a Coxeter group: Generators + quadratic and braid relations $\rightsquigarrow$ B the Artin-Tits group: Generators + braid relations

$\updownarrow$ (Brieskorn)

$$B = \pi_1(X/W)$$

General goal

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Why considering $B(e, e, n)$, the complex braid group attached to $G(e, e, n)$?

Garside structure

Let us recall the notion of a Garside structure.

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Let M be a monoid and $f, g \in M$.
We say that f left-divides g, written $f \preceq g$, if $fg' = g$ holds for some $g' \in M$.
Similarly, we can define the notion of right division.

Garside structure

Definition

We say that M is a Garside monoid if

- M is cancellative,
- there exists $\lambda : M \longrightarrow \mathbb{N}$ s.t. $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $g \neq 1 \Longrightarrow \lambda(g) \neq 0$,
- any two elements of M have a gcd and an lcm,
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A Garside group is the group of fractions of a Garside monoid.

Interval structure

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For $g \in G$, we define a monoid $M([1, g])$ by the presentation with

- Generators: A set $\underline{P}$ in bijection with the interval

$$[1, g] := \{x \in G \mid 1 \preceq x \preceq g\}$$

- Relations: $\underline{x}\ \underline{y} = \underline{z}$ if $xy = z$ and $x \preceq z \preceq g$, that is $\ell(x) + \ell(y) = \ell(z)$.

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Similarly, we can introduce the partial order relation $x \preceq_r y \Longleftrightarrow \ell(yx^{-1}) + \ell(x) = \ell(y)$, and the interval $[1, g]_r$ then define the monoid $M([1, g]_r)$.

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Theorem (Michel)

If $[1, g] = [1, g]_r$ and both posets $([1, g], \preceq)$ and $([1, g]_r, \preceq_r)$ are lattices, then $M([1, g])$ is a Garside monoid with Garside element $\underline{g}$ and with simples $\underline{P}$.

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Theorem

$B^+(W)$ is a Garside monoid with Garside element $\underline{w_0}$ and with simples $\underline{W}$.

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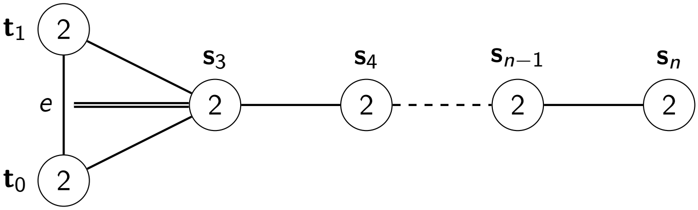

Presentation for $G(e,e,n)$

It is shown that this group is isomorphic to the complex reflection group $G(e,e,n)$ (Broué-Malle-Rouquier),

with $\mathbf{t}_i \longmapsto t_i := \begin{pmatrix} 0 & \zeta_e^{-i} & 0 \\ \zeta_e^{i} & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ with $i = 0,1$ and

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Coxeter cases $n = 2$ (dihedral groups), $e = 1$ (symmetric groups), $e = 2$ (type D_n).

Presentation of B-M-R for $B(e, e, n)$

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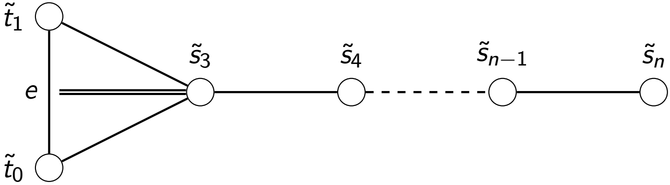

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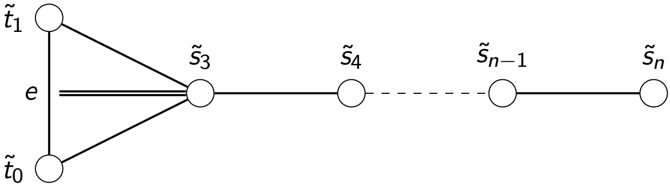

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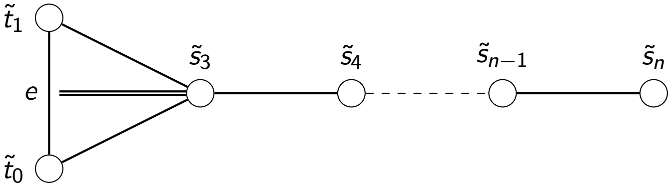

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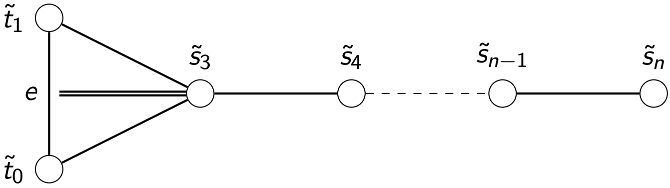

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If $e \geq 3$ and $n \geq 3$, the monoid defined by this presentation fails to embed in $B(e, e, n)$ (Corran)

Thus, this presentation does not give rise to a Garside structure for $B(e, e, n)$!

Presentation of C-P for $B(e, e, n)$

Let $\tilde{t}_i := \tilde{t}_{i-1}\tilde{t}_{i-2}\tilde{t}_{i-1}^{-1}$ for $2 \leq i \leq e-1$. Consider the following diagram presentation:

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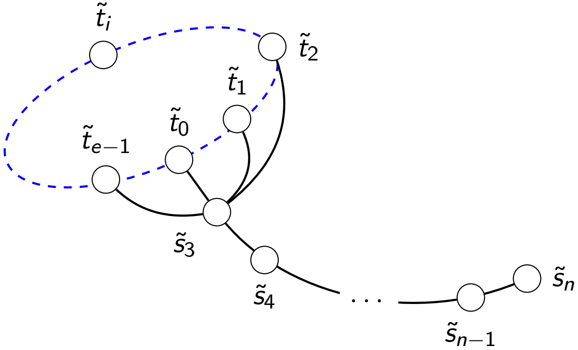

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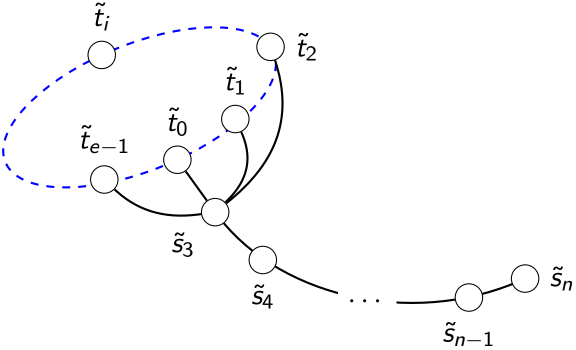

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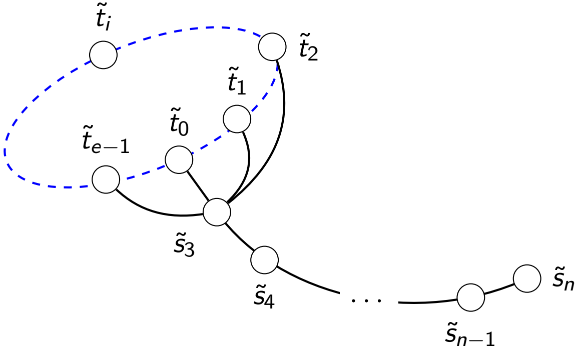

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This presentation gives rise to a Garside structure for $B(e, e, n)$!

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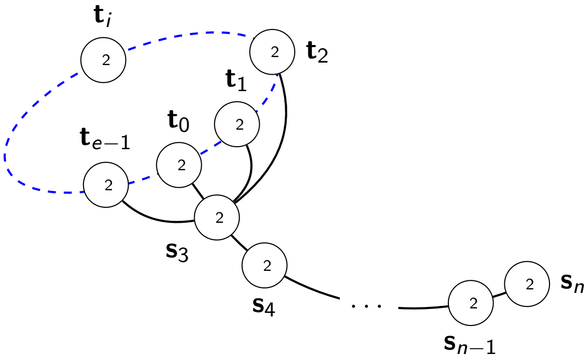
Adding $x^2 = 1$ for all generators x of the presentation of Corran-Picantin gives a presentation of a group isomorphic to $G(e, e, n)$



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the simples are precisely the elements of the form $\delta_2\delta_3\cdots\delta_n$ where δ_i is a divisor of Δ_i for $2 \leq i \leq n$.

Second homology group

Theorem (Callegaro,Marin)

Let $B := B(e, e, n)$ with $e \geq 2$ and $n \geq 2$.

- If $n = 2$, then $H_2(B, \mathbb{Z}) = 0$ if e is odd and $H_2(B, \mathbb{Z}) = \mathbb{Z}$ if e is even (Salvetti).
- If $n = 3$, then $H_2(B, \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z}$.
- If $n = 4$ and e is odd, then $H_2(B, \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- If $n = 4$ and e is even, then $H_2(B, \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$.
- If $n \geq 5$, then $H_2(B, \mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Parabolic submonoids

Let δ be a balanced element of $Div(\Delta)$. Denote by B_δ the subgroup of $B(e, e, n)$ generated by the elements of $\{\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_{e-1}, \tilde{s}_3, \cdots, \tilde{s}_n\}$ that are in $Div(\delta)$.

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We say that B_δ is a standard parabolic subgroup of $B(e, e, n)$ if $Div(\delta) = Div(\Delta) \cap B_\delta$ (definition of Godelle for standard parabolic subgroups in a Garside structure).

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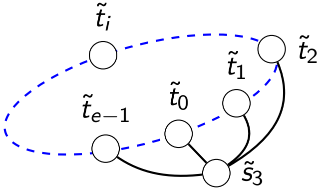
The balanced elements of $Div(\Delta)$ are the lcm of the subsets of $\{\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_{e-1}, \tilde{s}_3, \cdots, \tilde{s}_n\}$. Denote δ one of these balanced elements. We have B_δ a standard parabolic subgroup of $B(e, e, n)$ in the sense of Godelle.

Parabolic submonoids

For example, consider the presentation of Corran-Picantin for $B(e, e, 3)$



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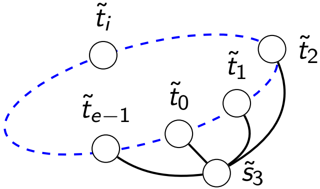

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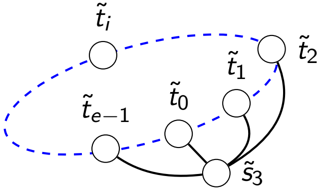

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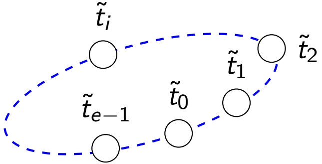

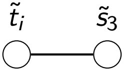

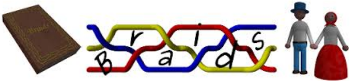

Thank you!