Le monoïde de 0-Rook, et autres monoïdes $\mathcal{J}$-triviaux

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Contents

Some basic definitions

Group **Monoid** **Semigroup**

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- Associative
- Has a unit
- Each element has an inverse

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- $\rho : G \to GL(V)$ group morphism
- if $\dim V < \infty$: $GL(V) \cong GL_n(\mathbb{C})$

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Group algebra $\mathbb{C}G$

Monoid algebra $\mathbb{C}M$

Preorders and equivalence relations

Definition (Green)

Let M a monoid, $x, y \in M$. We say :

- $x \leq_{\mathcal{R}} y$ iff $xM \subseteq yM$
- $x \leq_{\mathcal{L}} y$ iff $Mx \subseteq My$
- $x \leq_{\mathcal{J}} y$ iff $MxM \subseteq MyM$

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Warning : 1 on top!

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$\mathcal{K}$-triviality

Definition

A monoid is $\mathcal{K}$-trivial ($\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$) if all its $\mathcal{K}$-classes are of cardinality one.

Ordered monoids

Definition (Green)

Let M a monoid with $\leq$ a partial order. M is called :

- right ordered iff $xy \leq x$ for all $x, y \in M$.
- left ordered iff $yx \leq x$ for all $x, y \in M$.
- left-right ordered iff $xy \leq x$ and $yx \leq x$ for all $x, y \in M$.

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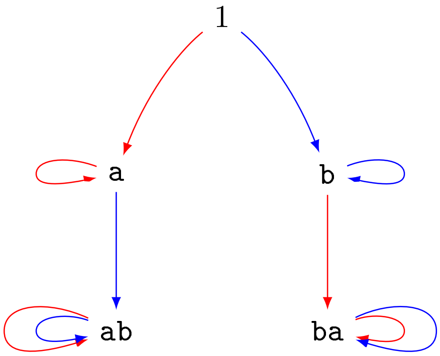
$\mathcal{K}$-triviality can be seen in Cayley graph

Example : Cayley-graph and $\mathcal{K}$-triviality

$M = \langle a, b \rangle$
$a^2 = a$; $b^2 = b$; $aba = ab$; $bab = ba$



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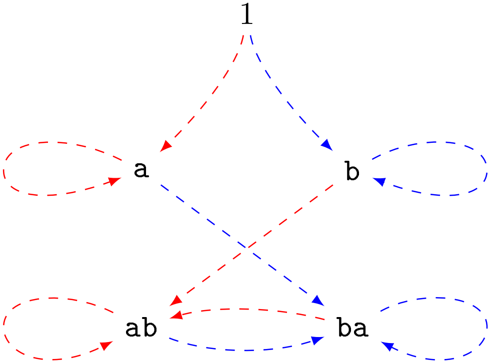

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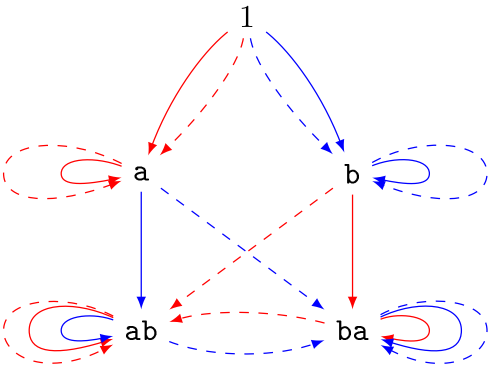

Aperiodic monoid

Let M a monoid, $x \in M$.

$$\begin{tikzcd}
 & & & & & \cdots \arrow[dl] & \\
x \arrow[r] & x^2 \arrow[r] & x^3 \arrow[r] & \cdots \arrow[r] & x^n \arrow[dr] & & x^{n+2} \arrow[ul] \\
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M is said *aperiodic* is $\forall x \in M, \exists n, x^n = x^{n+1}$.
Notation $x^\omega := x^n$ idempotent
$E(M)$: set of idempotents of M.

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$\mathcal{K}$-trivial $\Rightarrow$ aperiodic.

Idempotents and representation of finite semigroup

Let S a finite semigroup.

- $e \in S$ idempotent $\Rightarrow eSe$ monoid with identity e.
  G_e group of units.

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- If e, f idempotents. $e\mathcal{J}f \Rightarrow G_e \cong G_f$.
- A $\mathcal{J}$-class J is said *regular* if it contains an idempotent.
  If $e \in J$ then $G_e = eSe \cap J$.

Idempotents and representation of finite semigroup

Theorem (Clifford, Munn, Ponizovskii)

Let S a semigroup, K a field, $E = \{e_J\}$ an idempotent transversal over the regular $\mathcal{J}$-classes J of S. Let $G_J := G_{e_J}$.
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Remark : the aperiodic monoids is the class of monoids for which all G_J are trivial.

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Definition by presentation

$$\mathfrak{S}_n$$

Symmetric Group

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$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

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Iwahori-Hecke algebra

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Hecke monoid at $q = 0$

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Symmetric Group	Iwahori-Hecke algebra	Hecke monoid at $q = 0$
$s_i^2 = 1$	$T_i^2 = q1 + (q-1)T_i$	$\pi_i^2 = \pi_i$
$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$	$T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i$	$\pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i$
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$s_i = T_i$ $\qquad\qquad$ $\pi_i = T_i + 1$

Definition by action

Action on $\mathfrak{S}_n$
Bubble sort algorithm :

$$(s_1 \dots s_n) \cdot \pi_i = \begin{cases} s_1 \dots s_{i-1} s_{i+1} s_i s_{i+2} \dots s_n & \text{if } s_i < s_{i+1}, \\ s_1 \dots s_n & \text{otherwise.} \end{cases}$$

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$$3726145 \cdot \pi_5 = 3726415$$
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Theorems for H_n^0

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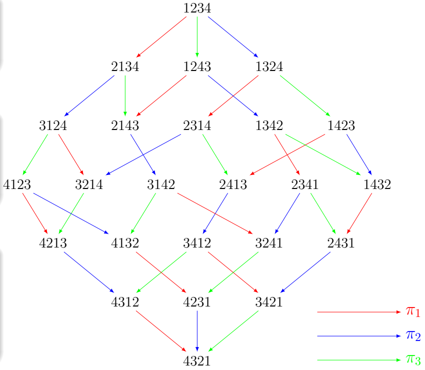

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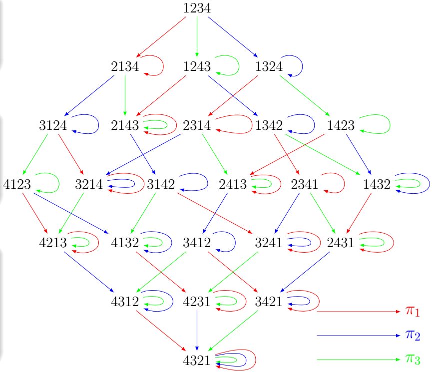

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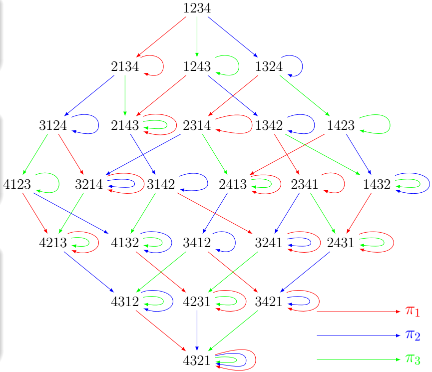
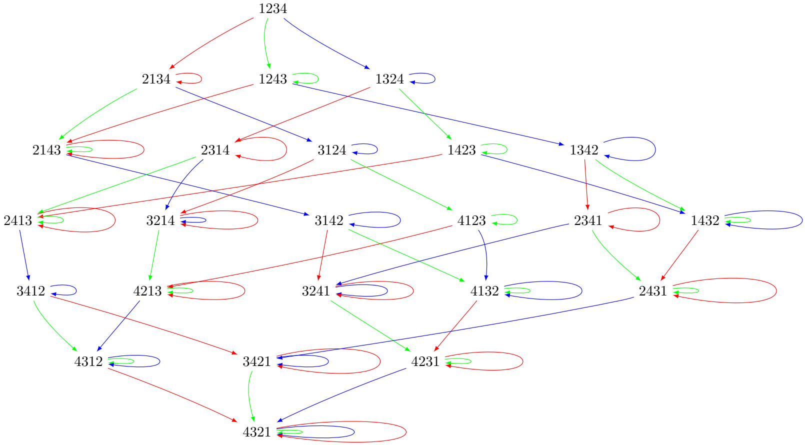
H_n^0 is $\mathcal{J}$-trivial



$\mathcal{J}$-triviality : H_n^0 right Cayley graph



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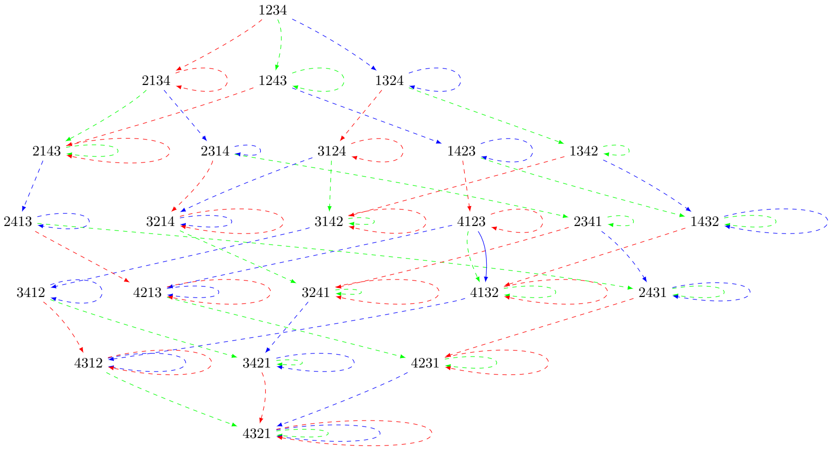
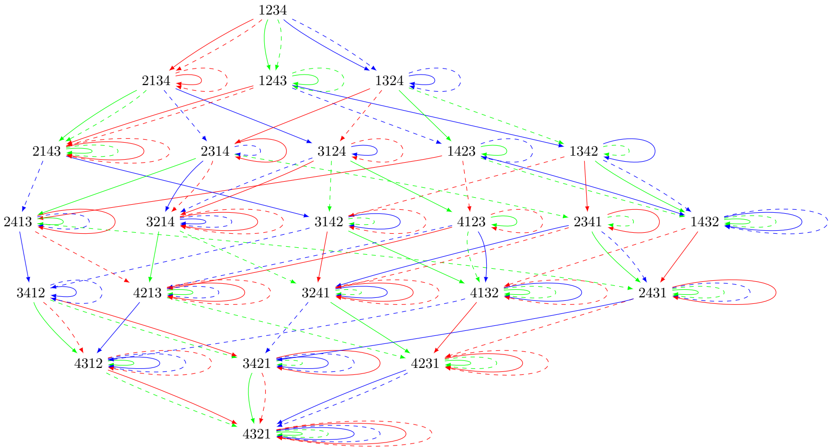


$\mathcal{J}$-triviality : H_n^0 bisided Cayley graph



Contents

Simple modules

M is $\mathcal{J}$-trivial and $x \in M$.

Definition

Let $S_x = \langle \epsilon_x \rangle$ vector space, right action of $y \in M$:

$$\epsilon_x \cdot y = \begin{cases} \epsilon_x & \text{if } xy = x \\ 0 & \text{otherwise.} \end{cases}$$

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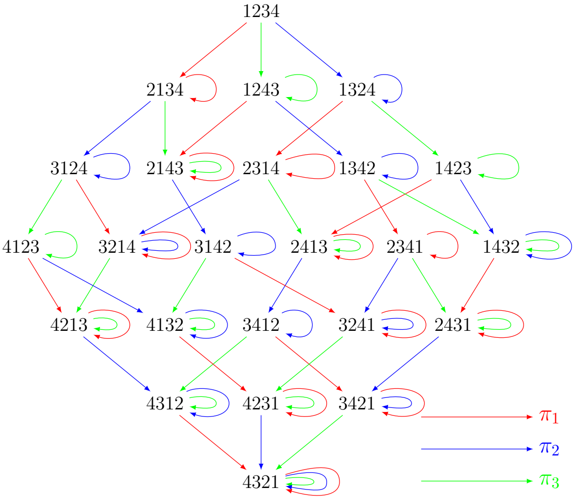
Theorem (Denton, Hivert, Schilling, Thiéry)

Any simple module is isomorphic to S_x for $x \in M$ (thus one-dimensional).

Simple modules



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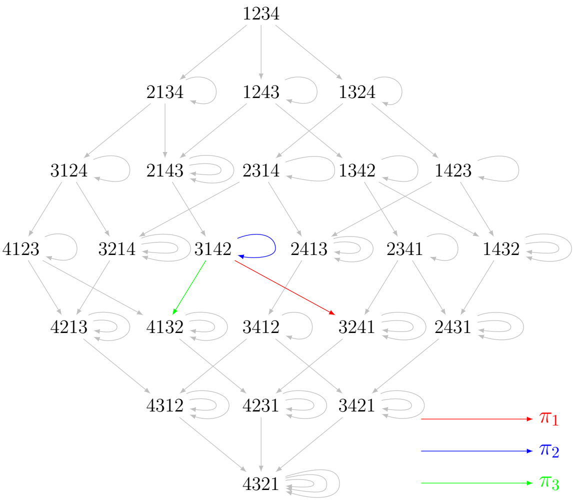

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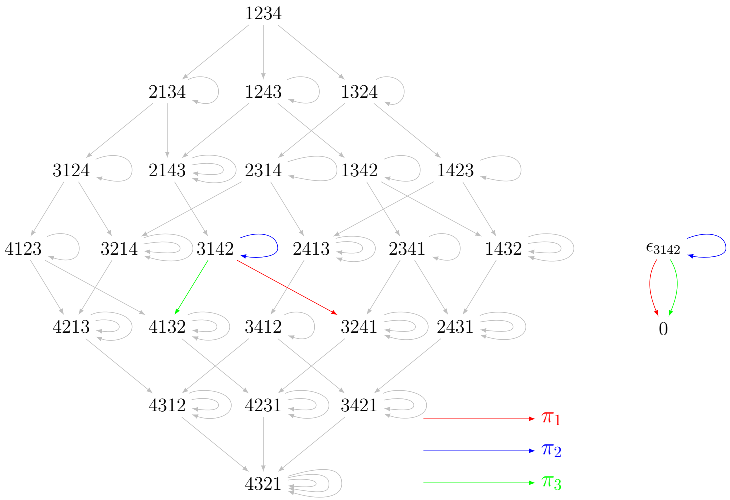
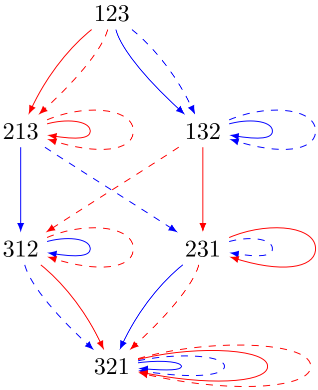


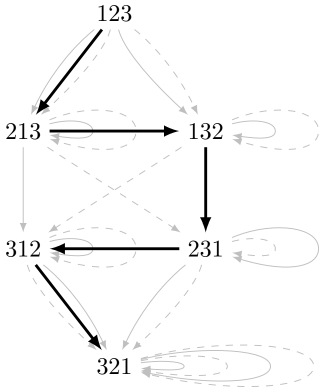
Simple modules : proof

- $\leq_{\mathcal{J}}$ partial order



Simple modules : proof

- $\leq_{\mathcal{J}}$ partial order
- $(x_1, \dots x_n)$ linear extension : $x_i \leq_{\mathcal{J}} x_j \Rightarrow i \leq j$

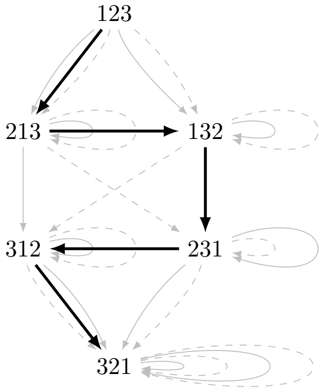


Simple modules : proof

- $\leq_{\mathcal{J}}$ partial order
- $(x_1, \dots x_n)$ linear extension : $x_i \leq_{\mathcal{J}} x_j \Rightarrow i \leq j$
- $F_i := \mathbb{K}\{x_j \mid j \leq i\}$, $F_0 := \{0_{\mathbb{K}}\}$:

$$F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n$$

composition series for F_n regular representation



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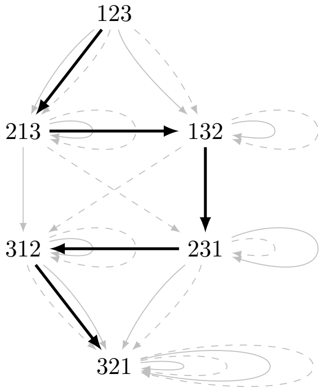
  composition series for F_n regular representation
- ${}^{F_i}\!/_{F_{i-1}} \cong S_{x_i}$



Simple modules

Theorem (Norton, Carter & Denton, Hivert, Schilling, Thiéry)

$(S_e)_{e\in E(M)}$ *complete set of pairwise non isomorphic simple modules*
Furthermore $\{x - x^\omega | x \notin E(M)\}$ *basis for* $\operatorname{rad}\mathbb{K}M$.

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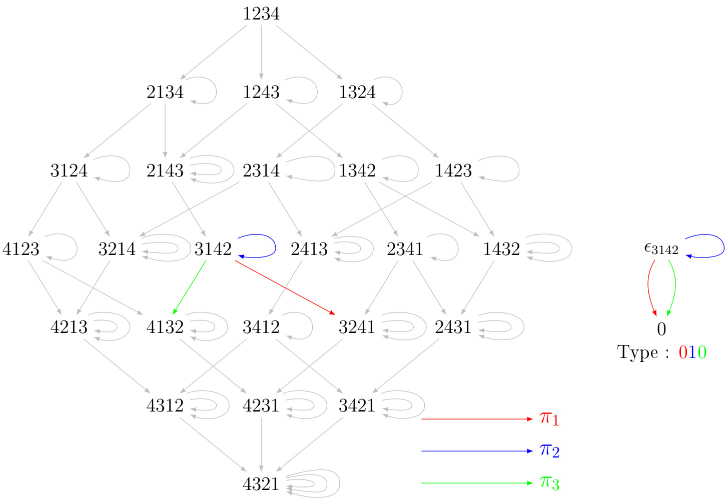
Remark :
$e \star f := (ef)^\omega$

$$\mathbb{K}M / \operatorname{rad}\mathbb{K}M \cong (\mathbb{K}E(M), \star)$$

Furthermore :

$$\begin{array}{ccc} \mathbb{K}M & \longrightarrow & \mathbb{K}M / \operatorname{rad}\mathbb{K}M \\ x & \longmapsto & x^\omega \end{array}$$

Simple modules



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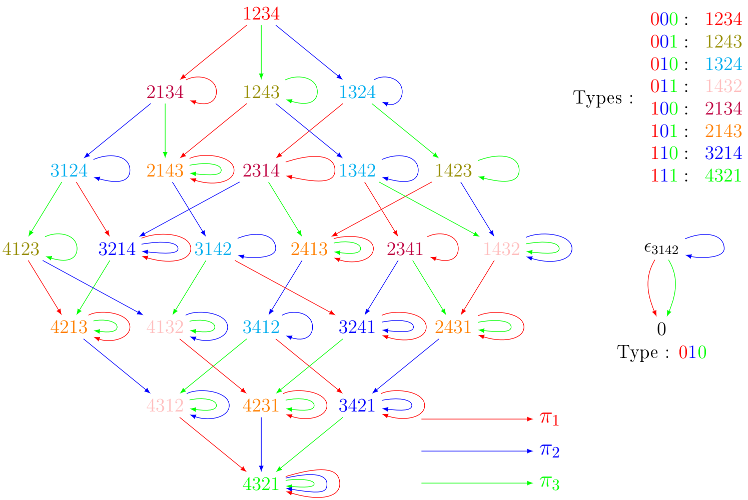
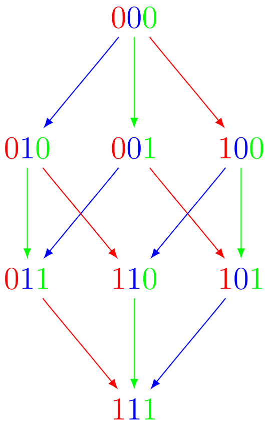


Simple modules

$E(H_4^0)$:



Projective indecomposable modules

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Remark : a $\mathcal{J}$-trivial monoid always have a zero element.

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For $x \in H_n^0$, we define its right descent set by

$$D_R(x) = \{1 \leq i \leq n-1 \mid x\pi_i = x\}.$$

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Example : positions

Let $x = 413256$.

Projective indecomposable modules

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Example : positions

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Notation :
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Projective indecomposable modules

Theorem (Norton, Carter & Denton, Hivert, Schilling, Thiéry)

$e \in E(M)$. *We set :*

$$L(e) := Me$$

$$L_=(e) := \{x \in Me \mid \mathrm{rfix}(x) = e\}$$

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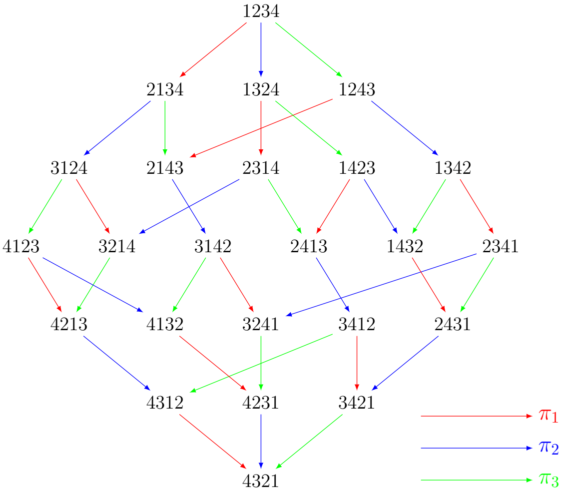
Then, the projective module P_e associated to S_e is isomorphic to $\mathbb{K}L(e) / \mathbb{K}L_<(e)$.

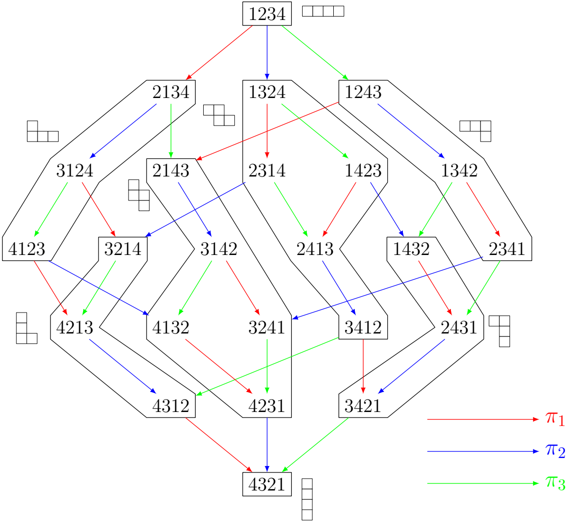
Basis of P_e : image of $L_=(e)$ in the quotient.

Action of $m \in M$ on $x \in L_=(e)$: $m \cdot x = \begin{cases} mx & \text{if } \mathrm{rfix}(mx) = e \\ 0 & \text{otherwise.} \end{cases}$

Projective indecomposable modules



Projective indecomposable modules



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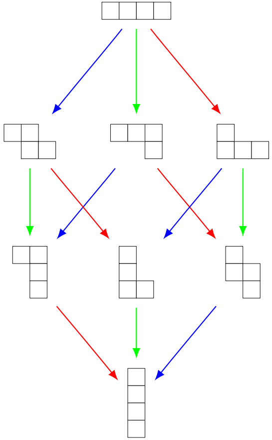

Effectivity

Simple modules, projective indecomposable modules, Cartan matrix, quiver.

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- Algebra of finite dimension :

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Contents

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Rook matrix of size n = set of non attacking rooks on an $n \times n$ matrix.

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$$\text{Rook Matrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & \text{♜} & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \text{♜} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{♜} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{♜} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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\text{Rook Vector} & 0 \quad 4 \quad 2 \quad 3 \quad 1 & 0 \quad 3 \quad 0 \quad 4 \quad 1
\end{array}
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\text{Rook Vector} & \begin{matrix} 0 & 4 & 2 & 3 & 1 \end{matrix} & \begin{matrix} 0 & 3 & 0 & 4 & 1 \end{matrix}
\end{array}
$$

The product of two rook matrices is a rook matrix.
Rook Monoid R_n = submonoid of the rook matrices

$$M_n \supset R_n \supset \mathfrak{S}_n$$

$$\mathfrak{S}_n$$

$$\mathfrak{S}_n \xleftarrow{q=1} \mathcal{H}_n(q)$$

Iwahori

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Iwahori , Solomon.

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\end{array}
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Definition by right action on R_n

Operators $\pi_0, \pi_1, \ldots \pi_{n-1}$ acting on rook vectors

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Bubble sort operators $\pi_1, \dots, \pi_{n-1}$:

$$(r_1 \dots r_n) \cdot \pi_i = \begin{cases} r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \dots r_n & \text{otherwise,} \end{cases}$$

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$$(r_1 \dots r_n) \cdot \pi_0 = \quad 0 r_2 \dots r_n.$$

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$$45321 \cdot \pi_1 = 54321$$
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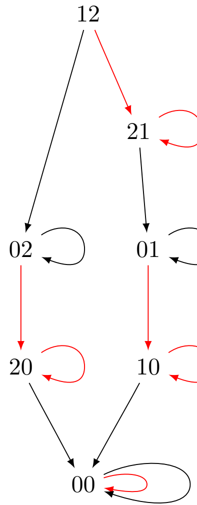

Definition by presentation

Generators : $\pi_0, \ldots, \pi_{n-1}$

Relations :

$$\begin{aligned} \pi_i^2 &= \pi_i & 0 \le i \le n-1, \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} & 1 \le i \le n-2, \\ \pi_i \pi_j &= \pi_j \pi_i & |i-j| > 1. \\ \pi_1 \pi_0 \pi_1 \pi_0 &= \pi_0 \pi_1 \pi_0 = \pi_0 \pi_1 \pi_0 \pi_1 & \end{aligned}$$

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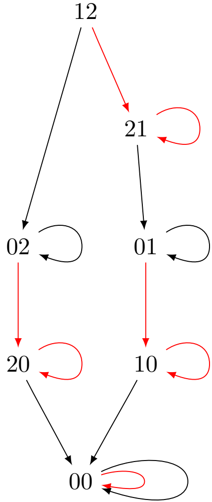

Theorem

Both definitions (presentation and action on R_n) are equivalent.

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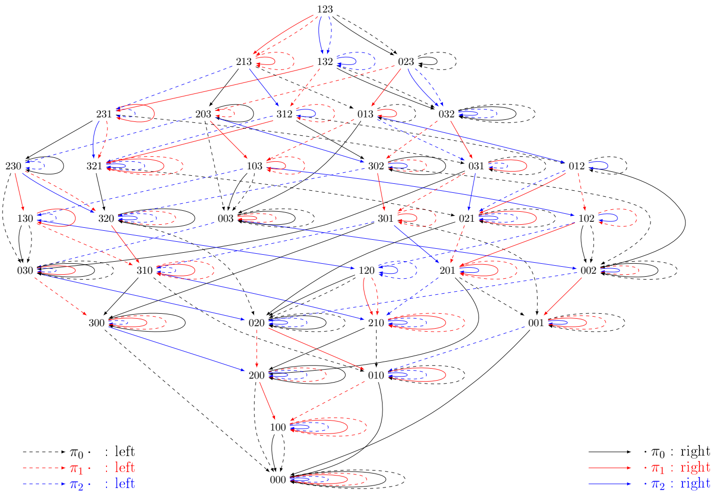
Key Fact :

Theorem

The map $f : \left| \begin{array}{ccc} R_n^0 & \longrightarrow & R_n \\ r & \longmapsto & \mathbf{1}_n \cdot r \end{array} \right.$ *is a bijection.*

$\mathcal{J}$-triviality : R_n^0 bisided Cayley graph



Simple modules

Theorem

R_n^0 is $\mathcal{J}$-trivial.

Simple modules

Theorem

R_n^0 is $\mathcal{J}$-trivial.

Corollary (Application of Denton-Hivert-Schilling-Thiéry)

R_n^0 has 2^n idempotents.
It has thus 2^n simple modules of dimension 1.

Descent set

Definition

For $\pi \in R_n^0$, we define its right R-descent set by

$$D_R(\pi) = \{0 \le i \le n-1 \mid \pi\pi_i = \pi\}.$$

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 & \boxed{2} & \boxed{3} & \\
 & & \boxed{0} & \\
 & & \boxed{0} & \boxed{7}
\end{array}$$

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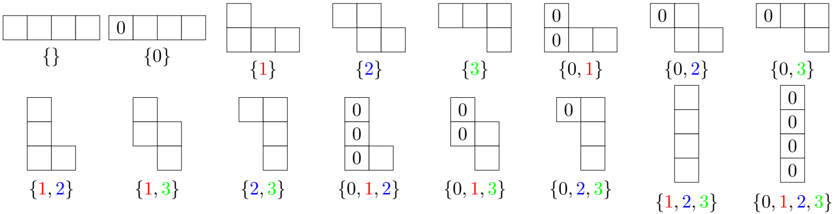
Warning : $\begin{array}{c} \boxed{0} \\ \boxed{0} \end{array}$ and not $\boxed{0}\boxed{0}$.

Descent class

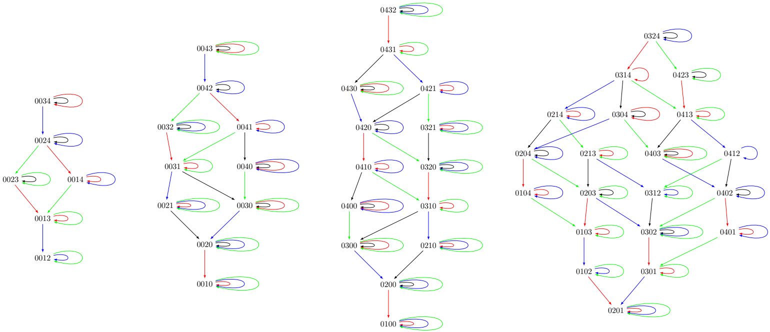
List of the R-descent types for R_4^0:



Projective modules

Theorem (Application of Denton-Hivert-Schilling-Thiéry)

The projective indecomposable R_n^0-modules are indexed by the R-descent type and isomorphic to the quotient of the associated R-descent class by the finer R-descent class.

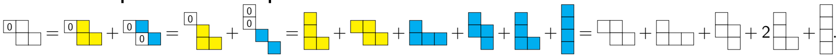


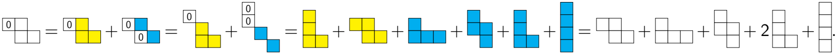
Projectivity over H_n^0

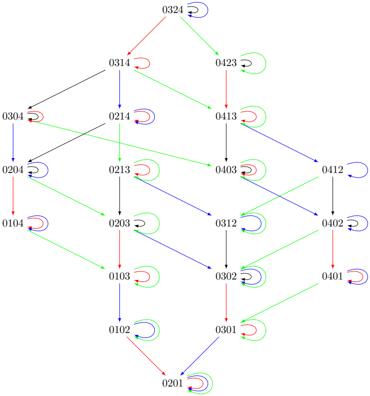
Theorem

The indecomposable projective R_n^0-module splits as a H_n^0-module as the direct sum of all the indecomposable projective H_n^0-modules whose descent classes are explicit.

Proof : explicit decomposition



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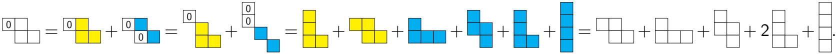

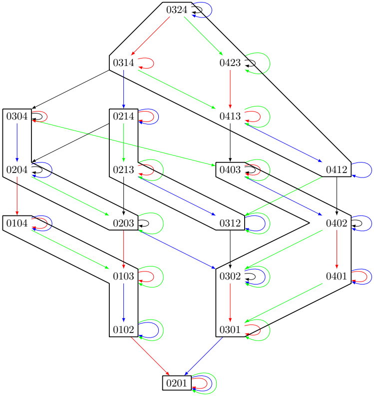
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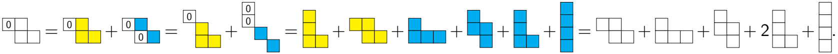

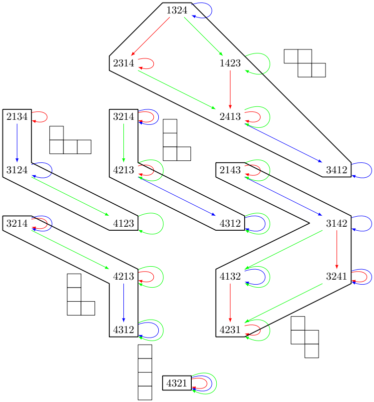
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Tower of monoids

$$R_1^0 \subset R_2^0 \subset \cdots \subset R_n^0 \subset \dots$$

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Results :

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Theorem

M irreducible algebraic monoid with a zero element.
M is regular iff $G(M)$ is reductive.

Renner monoids and Weyl groups

Definition

M regular irreducible algebraic monoid with a zero element.
T maximal Torus inside a Borel subgroup of $G(M)$, its normalizer $N_{G(M)}(T)$.

The Renner monoid $R(M)$ is $\overline{N_{G(M)}(T)} / T$.

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Example

$M = M_n$ then $R(M) = R_n$

Results

- Definition of 0-Renner monoids in type B, D
  - Presentation
  - Function on some rook vectors

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Example : type B_5

1_{10} :

$$\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1}\,1\,2\,3\,4\,5$$

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Example : type B_5

$1_{10} \cdot \pi_1\pi_3\pi_0$:

$$0\,0\,0\,0\,0\,\overline{1}\,2\,4\,3\,5$$

Results

- Definition of 0-Renner monoids in type B, D
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- Equivalence of the definitions

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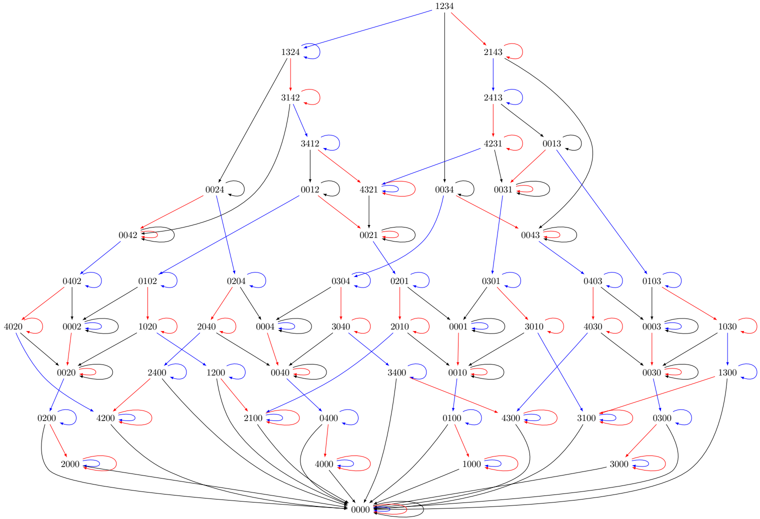
  - Presentation
  - Function on some rook vectors
- Equivalence of the definitions
- On going : $\mathcal{J}$-triviality

Results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2143
3142
2413
3412
4231
0013
0024
0012
4321
0034
0031
0042
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0402
0102
0204
0304
0201
0301
0403
0103
4020
0002
1020
2040
0004
3040
2010
0001
3010
4030
0003
1030
0020
2400
1200
0040
3400
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4200
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4300
3100
0300
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4000
1000
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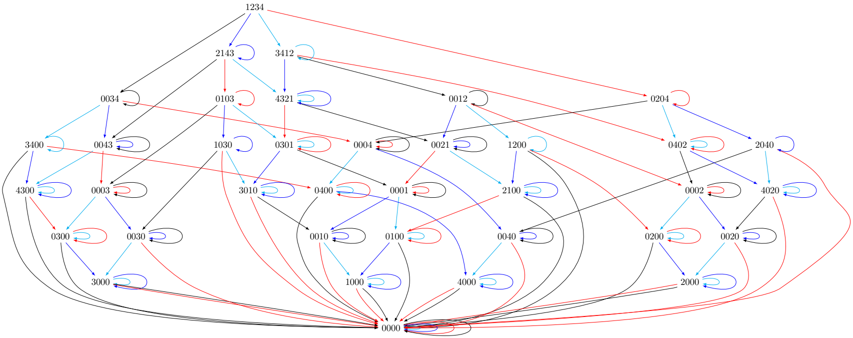
Results


1234
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Results

- Definition of 0-Renner monoids in type B, D
  - Presentation
  - Function on some rook vectors
- Equivalence of the definitions
- On going : $\mathcal{J}$-triviality
- On going : representation theory

THANK YOU FOR YOUR OUTSTANDING ATTENTION!!

Descent classes are not intervals